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Coupled Painlevé systems and quartic potentials

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Abstract

Similarity reductions of the Hirota–Satsuma system and another gauge-related system yield non-autonomous Hamiltonian systems with quartic potentials. We present classes of special solutions and Bäcklund transformations which are interpreted in terms of the action of an affine Weyl group on the space of parameters. Some other quartic oscillators related to coupled Painlevé-type equations are briefly considered. We show how separation of variables also has an application in this context.

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1. Introduction

Two-particle Hamiltonians with quartic potentials, of the form

$$H = \tfrac{1}{2}(p_1^2 + p_2^2) + aq_1^4 + bq_1^2q_2^2 + cq_2^4 \tag{1.1}$$

are known to be integrable in only four non-trivial cases:

(a) $a : b : c = 1 : 2 : 1$;
(b) $a : b : c = 1 : 12 : 16$;
(c) $a : b : c = 1 : 6 : 1$;
(d) $a : b : c = 1 : 6 : 8$.

The complete integrability of such systems, and the extension of (1.1) to include extra terms which preserve that property, has been considered from various points of view [1, 4, 16]. In particular, the authors of [1] showed how cases (c) and (d) (with the inclusion of some extra inverse square terms) could be derived as stationary flows of the Hirota–Satsuma system,

$$u_t = \tfrac{1}{2}u_{3x} + 3uu_x - 6\phi\phi_x \qquad \phi_t = -\phi_{3x} - 3u\phi_x \tag{1.2}$$

and another coupled KdV system,

$$\begin{aligned} f_t &= -f_{3x} - \tfrac{3}{2}ff_{xx} - \tfrac{3}{2}f_x^2 + \tfrac{3}{2}f^2f_x - 3fg_x - 3gf_x \\ g_t &= \tfrac{1}{4}(2g_{3x} + 12gg_x + 6fg_{xx} + 12gf_{xx} + 18f_xg_x - 6f^2g_x \\ &\quad +3f_{4x} + 3ff_{3x} + 18f_xf_{xx} - 6f^2f_{xx} - 6ff_x^2) \end{aligned} \tag{1.3}$$

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respectively; both systems are associated with fourth-order Lax operators. This extended the results of [6], where it was demonstrated that the integrable cases of the Hénon–Heiles system arise as stationary flows of integrable partial differential equations (PDEs) obtained from second- and third-order Lax operators.

The observation of this author in [8, 9] was that, by allowing scaling similarity reductions of the PDEs instead of stationary flows, the same Hénon–Heiles systems are obtained but with extra non-autonomous (time-dependent) terms appearing in the potential. The non-autonomous systems are equivalent to certain fourth-order ordinary differential equations (ODEs) of Painlevé type, which appear in a recent Painlevé classification of fourth- and fifth-order equations in the polynomial class made by Cosgrove [2]. Below we consider the scaling similarity reductions of (1.2) and (1.3), which (as outlined in [8]) yield non-autonomous extensions of cases (c) and (d) Hamiltonian systems (1.1). Coupled Painlevé equations corresponding to cases (a) and (b) are briefly discussed at the end, where we outline an application of separation of variables to the non-autonomous version of case (a).

Our main results here are the Bäcklund transformations (BTs) for the non-autonomous cases (c) and (d), and their interpretation in terms of the action of a subgroup of the affine Weyl group of the root system A_3 on the space of parameters. It turns out that for special parameter values these coupled Painlevé equations can be solved in terms of the second Painlevé transcendent PII. This was one of our motivations for studying these equations, since in his recent work [2] Cosgrove has found a new fifth-order Painlevé-type equation in the polynomial class, which also admits various particular solutions in terms of PII. This fifth-order equation admits a first integral of fourth order but of third degree, and is conjectured to be related to the Hirota–Satsuma system [3]. It is tempting to suggest that this fourth-order equation should be equivalent to the coupled pair of ODEs (2.1) constructed below, but as yet this connection remains elusive.

2. Coupled Painlevé equations

The scaling similarity solutions of (1.2) and (1.3) take the form

$$u(x,t) = (-t/3)^{-2/3}U(z) \qquad \phi(x,t) = (-t/3)^{-2/3}\Phi(z)$$

and

$$f(x,t) = (-t/3)^{-1/3}F(z) \qquad g(x,t) = (-t/3)^{-2/3}G(z)$$

where

$$z(x,t) = (-t/3)^{-1/3}x.$$

The derivation of the ODEs for these scaling similarity variables is very similar to the case of the stationary flows given in [1], so we omit the details here.

For the reduction of the Hirota–Satsuma system (1.2) we define new dependent variables L_1, L_2 according to

$$U = L_1 + L_2 + z \qquad \Phi = (L_1 - L_2)/2$$

and then we find that these variables satisfy the following coupled pair of second-order ODEs:

$$\begin{aligned} L_1L_1'' - \tfrac{1}{2}(L_1')^2 + (L_1 + 3L_2 + 2z)L_1^2 + \tfrac{1}{2}\ell_1^2 &= 0 \\ L_2L_2'' - \tfrac{1}{2}(L_2')^2 + (3L_1 + L_2 + 2z)L_2^2 + \tfrac{1}{2}\ell_2^2 &= 0 \end{aligned} \tag{2.1}$$

(ℓ_1 and ℓ_2 are constants, and $'$ denotes $\frac{\mathrm{d}}{\mathrm{d}z}$). The system (2.1) takes the form of two copies of the Painlevé equation P34 in Ince's classification [12] coupled together, and indeed the system reduces to that equation in the special cases $L_1 = 0 = \ell_1$ or $L_2 = 0 = \ell_2$ (which are discussed below).

For the scaling similarity reduction of the system (1.3) it is convenient to introduce the dependent variable

$$E = -G - \tfrac{1}{2}F' - \tfrac{1}{2}F^2 + z$$

and then we find the following coupled system for the variables E, F:

$$\begin{aligned} &EE'' - \tfrac{1}{2}(E')^2 - 2E^3 - \tfrac{3}{2}F^2E^2 + 2zE^2 + \tfrac{1}{2}\nu^2 = 0 \\ &F'' - 2F^3 - 3EF + 4zF - \xi = 0 \end{aligned} \tag{2.2}$$

(with ν and ξ being constants). The system (2.2) has the form of the second Painlevé equation PII coupled to P34; for $E = 0 = \nu$ it reduces to PII, while for $F = 0 = \xi$ it reduces to P34.

To make contact with the results of [1] for the stationary flows, we now present the Hamiltonian form of these coupled systems. Introducing the coordinates q_1, q_2 and Q_1, Q_2 given by

$$L_1 = q_1^2 \qquad L_2 = q_2^2 \qquad E = \tfrac{1}{4}Q_1^2 \qquad F = Q_2$$

we find that (2.1) becomes

$$\begin{aligned} &q_1'' + \frac{1}{2}\left(q_1^3 + 3q_2^2 q_1\right) + zq_1 + \frac{\ell_1^2}{4q_1^3} = 0 \\ &q_2'' + \frac{1}{2}\left(q_2^3 + 3q_1^2 q_2\right) + zq_2 + \frac{\ell_2^2}{4q_2^3} = 0 \end{aligned} \tag{2.3}$$

while (2.2) is equivalent to the system

$$\begin{aligned} &Q_1'' - \frac{1}{4}Q_1^3 - \frac{3}{4}Q_2^2 Q_1 + zQ_1 + \frac{4\nu^2}{Q_1^3} = 0 \\ &Q_2'' - 2Q_2^3 - \tfrac{3}{4}Q_1^2 Q_2 + 4zQ_2 - \xi = 0. \end{aligned} \tag{2.4}$$

Just as in the autonomous case [1], both systems (2.3) and (2.4) are Lagrangian, and may be derived from the Hamiltonians

$$h = \frac{1}{2}(p_1^2 + p_2^2) + \frac{1}{8}(q_1^4 + 6q_1^2 q_2^2 + q_2^4) + \frac{1}{2}z(q_1^2 + q_2^2) - \frac{1}{8}\left(\frac{\ell_1^2}{q_1^2} + \frac{\ell_2^2}{q_2^2}\right)$$

and

$$H = \frac{1}{2}(P_1^2 + P_2^2) - \frac{1}{16}(Q_1^4 + 6Q_1^2 Q_2^2 + 8Q_2^4) + \frac{1}{2}z(Q_1^2 + 4Q_2^2) - \frac{2\nu^2}{Q_1^2} - \xi Q_2$$

respectively (where we have introduced the canonical conjugate momenta $p_j = q_j'$, $P_j = Q_j'$, $j = 1, 2$). The Hamiltonians h and H may be expressed as logarithmic derivatives of tau-functions for these systems, but we shall not exploit this fact further here.

3. Bäcklund transformations

The key to finding the BTs for the systems (2.1) and (2.2) is to use the fact that the original PDE systems (1.2) and (1.3) are connected by a gauge transformation, and are related by a

Miura map to the same modified PDE. More precisely, the Hirota–Satsuma system may be obtained from a fourth-order Lax operator which factorizes as

$$(\partial - v_1)(\partial + v_1)(\partial + v_2)(\partial - v_2) \tag{3.1}$$

while the coupled PDE system (1.3) is obtained from the operator $(\partial+v_1)(\partial+v_2)(\partial-v_2)(\partial-v_1)$; each system of PDEs has a Miura map to the same modified PDE in the variables v_1, v_2 (this is described explicitly in [1]). Writing the Lax pairs for these systems in matrix form yields $sl(4)$ Lax matrices, and under the scaling similarity reduction it is straightforward to find isomonodromic Lax pairs for the systems (2.1) and (2.2), e.g. by the methods of [5]. The fact that the Lax matrices lie in $sl(4)$ suggests that the BTs for these systems should be understood in terms of the action of a subgroup of the affine Weyl group of the root system A_3.

In order to describe the BTs it is convenient to introduce auxiliary variables (corresponding to the modified variables v_1, v_2 in the PDE setting) and rewrite the coupled second-order ODEs as systems of first-order ODEs. The analogues of v_1, v_2 are

$$X_1 = \frac{L_1' + \ell_1}{2L_1} \qquad X_2 = \frac{L_2' + \ell_2}{2L_2} \tag{3.2}$$

and with these new dependent variables we find that (2.1) is equivalent to the first-order system

$$\begin{aligned} X_1' &= -\tfrac{1}{2}L_1 - \tfrac{3}{2}L_2 - X_1^2 - z \\ X_2' &= -\tfrac{3}{2}L_1 - \tfrac{1}{2}L_2 - X_2^2 - z \\ L_1' &= 2L_1X_1 - \ell_1 \\ L_2' &= 2L_2X_2 - \ell_2. \end{aligned} \tag{3.3}$$

Similarly, by introducing extra variables J, K we are able to rewrite (2.2) as the first-order system

$$\begin{aligned} J' &= E + \tfrac{3}{4}F^2 - J^2 - z \\ F' &= -K - F^2 + 2z \\ E' &= 2EJ - \nu \\ K' &= \left(2K - \tfrac{3}{4}E\right)F - \xi + 2. \end{aligned} \tag{3.4}$$

Our derivation of the BTs proceeds by making use of discrete symmetries and a one-to-one correspondence between solutions of the systems (3.3) and (3.4), inherited from the gauge relation between the original PDEs (1.2) and (1.3). The correspondence may be stated thus: given a solution of (3.3) with parameters ℓ_1, ℓ_2, a corresponding solution of (3.4) is given by

$$\begin{aligned} &J = -\tfrac{1}{2}(X_1 + X_2) \qquad F = X_2 - X_1 \\ &E = L_1 + L_2 + 2X_1X_2 + 2z \qquad K = L_1 - L_2 + 2X_1(X_2 - X_1) + 2z \end{aligned} \tag{3.5}$$

with the parameters related by

$$\xi = \ell_1 - \ell_2 \qquad \nu = \ell_1 + \ell_2 - 2.$$

Conversely, a solution of (3.3) is obtained from a solution of (3.4) by the inverse relations

$$\begin{aligned} &X_1 = -\tfrac{1}{2}F - J \qquad X_2 = \tfrac{1}{2}F - J \\ &L_1 = \tfrac{1}{2}(E + K) + \tfrac{3}{4}F^2 + JF - J^2 - 2z \qquad L_2 = \tfrac{1}{2}(E - K) - \tfrac{1}{4}F^2 - JF - J^2. \end{aligned} \tag{3.6}$$

In the variables p_j, q_j and P_j, Q_j this correspondence defines the analogue of the canonical transformation found in [1] for the autonomous case, so that the canonical 2-form is preserved i.e. $\sum_{j=1,2} \mathrm{d}p_j \wedge \mathrm{d}q_j - \mathrm{d}h \wedge \mathrm{d}z = \sum_{j=1,2} \mathrm{d}P_j \wedge \mathrm{d}Q_j - \mathrm{d}H \wedge \mathrm{d}z$.

Bearing this equivalence in mind it is sufficient to describe the action of the BTs on the system (3.3), since this will induce corresponding transformations on the system (3.4). Although the BTs are naturally described in terms of the affine Weyl group of A_3, the space of parameters (ℓ_1, ℓ_2) for the system (3.3) is only two dimensional, since the system derives from a fourth-order Lax operator of the particular form (3.1). Thus we expect our parameter space to be spanned by two simple roots in A_3. It turns out that only the orthogonal roots α_1 and α_3 are relevant here, while the part of the Weyl group involving the root α_2 is suppressed (see, for example, [11] for the standard notation for root systems). With a convenient choice of normalization we associate the root α_1 with the point $(2, 2)$ in the parameter space, while α_3 is associated with $(-2, 2)$. The simplest BT is just the reflection in the α_1 direction, which corresponds to a trivial symmetry of the system (3.3):

$$R_{\alpha_1}\colon \quad (\ell_1, \ell_2) \to (\ell_2, \ell_1) \qquad X_1 \leftrightarrow X_2 \qquad L_1 \leftrightarrow L_2. \tag{3.7}$$

Another simple property of the system (3.3) is that given solutions L_1, L_2 to the second-order system (2.1) (which depend only on the squares of the parameters ℓ_1, ℓ_2) we can define alternative modified variables

$$X_1^\dagger = \frac{L_1' - \ell_1}{2L_1} \qquad X_2^\dagger = \frac{L_2' - \ell_2}{2L_2}$$

and replace either X_1 by $X_1^\dagger$ or X_2 by $X_2^\dagger$, or both, in the first-order system (sending $\ell_1 \to -\ell_1$ or $\ell_2 \to -\ell_2$ where necessary). This yields two more BTs corresponding to reflections in the α_3 and $\alpha_1 + \alpha_3$ directions:

$$R_{\alpha_3}\colon \quad (\ell_1, \ell_2) \to (-\ell_2, -\ell_1) \qquad X_1 \to X_2^\dagger = X_2 - \frac{\ell_2}{L_2}$$

$$X_2 \to X_1^\dagger = X_1 - \frac{\ell_1}{L_1} \qquad L_1 \leftrightarrow L_2$$

$$R_{\alpha_1+\alpha_3}\colon \quad (\ell_1, \ell_2) \to (-\ell_1, \ell_2) \qquad X_1 \to X_1^\dagger = X_1 - \frac{\ell_1}{L_1}$$

$$X_2 \to X_2 \qquad L_1 \to L_1 \qquad L_2 \to L_2.$$

By combining these three reflectional symmetries we see that a generic point in the parameter space is connected to seven others, i.e. the points $(\pm\ell_1, \pm\ell_2)$ and $(\pm\ell_2, \pm\ell_1)$ are all connected by these reflections.

Having described the reflectional symmetries, we can now present the appropriate affine part of the Weyl group. In order to obtain a BT which induces a shift in the parameter space, we make use of the correspondence with the system (3.4) together with a discrete symmetry of that system, which may be described thus: given a solution (J, F, E, K) of the system (3.4) for parameter values ν, ξ, a solution for parameters $-\nu, -\xi$ is given by

$$J^\dagger = J - \frac{\nu}{E} \qquad F^\dagger = -F \qquad E^\dagger = E \qquad K^\dagger = -K - 2F^2 + 4z. \tag{3.8}$$

It is convenient to introduce the following quantities:

$$\hat{J} = -\tfrac{1}{2}(X_1^\dagger + X_2^\dagger) - \nu(L_1 + L_2 + 2X_1^\dagger X_2^\dagger + 2z)^{-1} \qquad \hat{F} = X_2^\dagger - X_1^\dagger$$

$$\hat{E} = L_1 + L_2 + 2X_1^\dagger X_2^\dagger + 2z \qquad \hat{K} = L_1 - L_2 + 2X_1^\dagger(X_2^\dagger - X_1^\dagger) + 2z.$$

Combining the discrete symmetry (3.8) with the correspondence (3.5) and (3.6) we find the BT for the system (3.3) equivalent to a shift in direction α_1 in the parameter space:

$$
\begin{aligned}
T_{\alpha_1}\colon\quad (\ell_1,\ell_2)\to(\ell_1+2,\ell_2+2)\qquad & X_1\to\overline{X}_1 = X_2^\dagger - \frac{(\ell_1+\ell_2+2)}{\hat{E}}\\
& X_2\to\overline{X}_2 = X_1^\dagger - \frac{(\ell_1+\ell_2+2)}{\hat{E}}\\
& L_1\to\overline{L}_1 = \tfrac{1}{2}(\hat{E}-\hat{K}) - \tfrac{1}{4}\hat{F}^2 - \hat{J}\hat{F} - \hat{J}^2\\
& L_2\to\overline{L}_2 = \tfrac{1}{2}(\hat{E}+\hat{K}) + \tfrac{3}{4}\hat{F}^2 + \hat{J}\hat{F} - \hat{J}^2 - 2z.
\end{aligned}
\tag{3.9}
$$

To express the inverse transformation it is convenient to use the quantities $J^\dagger$, $F^\dagger$, $E^\dagger$, $K^\dagger$ obtained from (3.8) via the correspondence (3.5), and define

$$
\begin{aligned}
\underline{L}_1 &= \tfrac{1}{2}(E^\dagger + K^\dagger) + \tfrac{3}{4}F^{\dagger 2} + J^\dagger F^\dagger - J^{\dagger 2} - 2z\\
\underline{L}_2 &= \tfrac{1}{2}(E^\dagger - K^\dagger) - \tfrac{1}{4}F^{\dagger 2} - J^\dagger F^\dagger - J^{\dagger 2}.
\end{aligned}
$$

Then the inverse of (3.9) may be given in the form

$$
\begin{aligned}
T_{-\alpha_1}\colon\quad (\ell_1,\ell_2)\to(\ell_1-2,\ell_2-2)\qquad & X_1\to\underline{X}_1 = -\frac{1}{2}F^\dagger - J^\dagger + \frac{(\ell_1-2)}{\underline{L}_1}\\
& X_2\to\underline{X}_2 = \frac{1}{2}F^\dagger - J^\dagger + \frac{(\ell_2-2)}{\underline{L}_2}\qquad L_1\to\underline{L}_1\qquad L_2\to\underline{L}_2.
\end{aligned}
$$

All possible BTs of the system (3.3) (and of the associated systems (2.1), (2.2) and (3.4)) should be obtained as combinations of the transformations presented above.

4. Special solutions

We have already remarked that the second-order systems (2.1) and (2.2) admit particular solutions in terms of solutions of the equation P34 or PII (whose solutions are themselves related by a one-to-one correspondence). By repeated application of the reflectional (R type) and translational (T type) BTs described above we obtain parameter families of special solutions to the system (3.3) on three families of lines in the (ℓ_1,ℓ_2) parameter space, denoted $\mathcal{L}_j$, $j=1,2,3$. Within each family $\mathcal{L}_j$, every line is related to every other by combinations of (3.7), (3.9) and the other symmetries.

The following families of lines are special:

- $\mathcal{L}_1$. Along the lines

  $$\ell_1 = 2n \quad\text{and}\quad \ell_2 = 2n \qquad n\in\mathbb{Z}$$

  there is a three-parameter family of special solutions to the system (3.3). These are obtained by starting from a point $(\ell,0)$ on the line $\ell_2=0$ and taking the special solution

  $$L_1 = L \qquad L_2 = 0 \qquad X_1 = \frac{L'+\ell}{2L} \qquad X_2 = Y$$

  where L is a solution of the equation P34 in the form

  $$LL'' - \tfrac{1}{2}(L')^2 + (L+2z)L^2 + \tfrac{1}{2}\ell^2 = 0$$

  and Y is a solution of the Riccati equation

  $$Y' + Y^2 = -\tfrac{3}{2}L - z.$$

(Note that (3.2) breaks down here.) The Riccati equation is linearized by the substitution $Y = (\log\psi)'$, where ψ is a solution of the generalized Lamé equation

$$\psi'' + V\psi = 0 \qquad V = \tfrac{3}{2}L + z$$

(with the usual elliptic potential replaced by a solution of P34). The solution of P34 depends on two arbitrary parameters, while the solution of the Riccati equation introduces another parameter. Thus by application of the BTs to any such solution on the line $\ell_2 = 0$ we obtain a three-parameter special solution at any point on the lines $\mathcal{L}_1$.

- $\mathcal{L}_2$. On the lines

$$\ell_1 \pm \ell_2 = 4n \qquad n \in \mathbb{Z}$$

there is a two-parameter family of special solutions, obtained by application of the BTs to the special solution

$$L_1 = L_2 = L \qquad X_1 = X_2 = \frac{L' + \ell}{2L}$$

at the point (ℓ, ℓ) on the line $\ell_1 - \ell_2 = 0$, where L satisfies P34 in the form

$$LL'' - \tfrac{1}{2}(L')^2 + (4L + 2z)L^2 + \frac{1}{2}\ell^2 = 0.$$

It is interesting to observe that the application of the BT (3.9) to the special solutions on this line is equivalent to two applications of the standard BT for P34 ($\ell \to \ell + 2$), with the quantity $\hat{E}$ being a solution of P34 at the intermediate parameter value $(\ell + 1)$.

- $\mathcal{L}_3$. On the lines

$$\ell_1 \pm \ell_2 = 2(2n + 1) \qquad n \in \mathbb{Z}$$

there is a three-parameter family of special solutions, generated by application of the BTs to a special solution of the system (3.3) at a point $(\ell, 2 - \ell)$ on the line $\ell_1 + \ell_2 = 2$. This special solution is obtained from the $E = 0 = \nu$ solution of the system (2.2), taking

$$L_1 = \tfrac{1}{2}K + \tfrac{3}{4}F^2 + JF - J^2 - 2z$$
$$L_2 = -\tfrac{1}{2}K - \tfrac{1}{4}F^2 - JF - J^2$$
$$X_{1,2} = \mp\tfrac{1}{2}F - J$$

with F a solution of the equation PII in the form

$$F'' - 2F^3 + 4zF - 2\ell + 2 = 0$$

the quantity K is given by

$$K = -F' - F^2 + 2z$$

while J is a solution of the Riccati equation

$$J' + J^2 = \tfrac{3}{4}F^2 - z.$$

(Once again the Riccati equation is solved in terms of a generalized Lamé equation.) Note that although the translational BT $T_{-\alpha_1}$ is not well defined for $E = 0 = \nu$, because the expression (3.8) for $J^\dagger$ breaks down, it may be applied consistently by setting $J^\dagger = J$ in this case.

Apart from these special solutions, the general solution of the system (3.3) may be obtained at the points

$$\ell_1 + \ell_2 = 4\,m \qquad \ell_1 - \ell_2 = 4n \qquad (m, n) \in \mathbb{Z}^2$$

(i.e. the intersection points of the lines $\mathcal{L}_2$) in terms of two copies of the equation PII. To see this we start from the point $(0, 0)$ in the parameter space and consider the Newton equations (2.3) for the Hamiltonian h. If we take the same separation variables λ_+, λ_- as in the autonomous case, so that $\lambda_\pm = \frac{1}{2}(q_1 \pm q_2)$, then these variables satisfy the decoupled equations

$$\lambda_\pm'' + 2\lambda_\pm^3 + z\lambda_\pm = 0.$$

Thus we see that λ_+, λ_- are just two independent solutions of the equation PII with zero parameter, and the general solution of (3.3) at $(0, 0)$ is given by

$$L_1 = (\lambda_+ + \lambda_-)^2 \qquad L_2 = (\lambda_+ - \lambda_-)^2 \qquad X_1 = \frac{\lambda_+' + \lambda_-'}{\lambda_+ + \lambda_-} \qquad X_2 = \frac{\lambda_+' - \lambda_-'}{\lambda_+ - \lambda_-}.$$

The formulae for the X_j break down in the special cases $\lambda_+ = \pm\lambda_-$, when these variables are found by solving a Riccati equation. The existence of these special solutions is strongly suggestive of a connection with the fifth-order equation Fif-IV in Cosgrove's classification [2].

We also note that, as a particular case of the special solutions on the lines $\mathcal{L}_2$, rational solutions of (3.3) may be found at the points $(\ell_1, \ell_2) = (2(m + n) \pm \frac{1}{2}, 2(m - n) \pm \frac{1}{2})$ for $(m, n) \in \mathbb{Z}^2$, by applying the BTs to the known rational solutions of P34. At the points where the families of lines $\mathcal{L}_j$ intersect there are also particular solutions expressed as mixtures of rational and Airy functions and their derivatives.

5. More coupled Painlevé equations

The case (a) and (b) Hamiltonians (1.1) also have non-autonomous extensions, equivalent to coupled pairs of Painlevé equations. For case (a) consider the system

$$\begin{aligned} &AA'' - \tfrac{1}{2}(A')^2 - 2(2\,A + 2B + z + a)A^2 + \tfrac{1}{2}\alpha^2 = 0 \\ &BB'' - \tfrac{1}{2}(B')^2 - 2(2\,A + 2B + z + b)B^2 + \tfrac{1}{2}\beta^2 = 0 \end{aligned} \tag{5.1}$$

where a, b, α, β are parameters. Observe that this system has special solutions in terms of P34 when $A = 0 = \alpha$ or $B = 0 = \beta$. If we make a change of variables

$$A = q_1^2 \qquad B = q_2^2$$

then it is straightforward to see that the equations for the coordinates q_1, q_2 are generated by the Hamiltonian

$$H = \frac{1}{2}(p_1^2 + p_2^2) - \frac{1}{2}(q_1^2 + q_2^2)^2 - \frac{1}{2}(z + a)q_1^2 - \frac{1}{2}(z + b)q_2^2 - \frac{1}{8}\left(\frac{\alpha^2}{q_1^2} + \frac{\beta^2}{q_2^2}\right) \tag{5.2}$$

and thus we see that (5.1) is just equivalent to a non-autonomous two-particle Garnier system. In the special case $a = b = \alpha = \beta = 0$ this system has appeared as a double-scaling limit of a Hermitian matrix model with a double-well ϕ^4 potential [15].

Another coupled Painlevé system,

$$\begin{aligned} &LL'' - \tfrac{1}{2}(L')^2 + 2\big(L - \tfrac{3}{4}M^2 + 2kM + 2z - 4k^2\big)L^2 + \tfrac{1}{2}\ell^2 = 0 \\ &M'' - 2M^3 + 6LM - 8kL + 8zM + m = 0 \end{aligned} \tag{5.3}$$

with parameters k, ℓ, m, may be obtained as a scaling similarity reduction of a higher-order member of the classical Boussinesq hierarchy (see, e.g., [7] for details of this hierarchy). Since the classical Boussinesq equation has a scaling similarity reduction to the fourth Painlevé transcendent PIV, the system (5.3) may naturally be considered as the next member in a PIV hierarchy (for a description of some ODE hierarchies see, e.g., [9, 14]). Note that for $L = 0 = \ell$ it has solutions in terms of PII, while for $M = 0 = m$ it has solutions in terms of P34. If we set

$$L = -\tfrac{1}{8}Q_1^2 \qquad M = Q_2$$

then the Newton equations for the coordinates Q_1, Q_2 are generated by the Hamiltonian

$$\begin{aligned} H = {} & \tfrac{1}{2}(P_1^2 + P_2^2) - \tfrac{1}{32}(Q_1^4 + 12Q_1^2Q_2^2 + 16Q_2^4) + kQ_1^2Q_2 \\ & +(z - 2k^2)Q_1^2 + 4zQ_2^2 - \frac{8\ell^2}{Q_1^2} + mQ_2 \end{aligned}$$

which is a non-autonomous extension of the case (b) Hamiltonian (1.1).

We have seen in the previous section that separation variables can be used to obtain special solutions even in the non-autonomous case, but they have other uses. As an example consider the non-autonomous Garnier system with Hamiltonian (5.2), which can be expressed as a sum of two Poisson commuting quantities:

$$\begin{aligned} H &= H_a + H_b \qquad \{H_a, H_b\} = 0 \\ H_a &= \frac{\mathcal{J}}{(a-b)} + \frac{1}{2}p_1^2 - \frac{1}{2}q_1^4 + \frac{b}{(a-b)}q_1^2q_2^2 - \frac{1}{2}(z+a)q_1^2 - \frac{\alpha^2}{8q_1^2} \\ H_b &= \frac{\mathcal{J}}{(b-a)} + \frac{1}{2}p_2^2 - \frac{1}{2}q_2^4 + \frac{a}{(b-a)}q_1^2q_2^2 - \frac{1}{2}(z+b)q_2^2 - \frac{\beta^2}{8q_2^2} \\ \mathcal{J} &= \frac{1}{2}(p_1q_2 - p_2q_1)^2 - \frac{1}{2}(a+b)q_1^2q_2^2 + \frac{1}{8}\left(\frac{\alpha^2q_2^2}{q_1^2} + \frac{\beta^2q_1^2}{q_2^2}\right). \end{aligned}$$

Since H_a and H_b both commute with the Hamiltonian we see immediately that

$$H_a' = -\tfrac{1}{2}q_1^2 \qquad H_b' = -\tfrac{1}{2}q_2^2. \tag{5.4}$$

The separation variables (elliptic coordinates) $\lambda_\pm$ are given by the zeros of

$$P(\zeta) = \frac{q_1^2}{(\zeta - a)} + \frac{q_2^2}{(\zeta - b)} - 1$$

so from $P(\lambda_\pm) = 0$ we have

$$q_1^2 = (b-a)^{-1}(\lambda_+ - a)(\lambda_- - a) \qquad q_2^2 = (a-b)^{-1}(\lambda_+ - b)(\lambda_- - b).$$

Introducing a generating function $G(p_j, \lambda_\pm)$ we find the conjugate momentum variables $\mu_\pm$ from

$$G = p_1\sqrt{\frac{(\lambda_+ - a)(\lambda_- - a)}{(b-a)}} + p_2\sqrt{\frac{(\lambda_+ - b)(\lambda_- - b)}{(a-b)}} \qquad \mu_\pm = \frac{\partial G}{\partial \lambda_\pm}.$$

It is then straightforward to show that

$$\mu_\pm^2 = \frac{\lambda_\pm}{4} + \frac{z}{4} + \frac{H_a}{2(\lambda_\pm - a)} + \frac{H_b}{2(\lambda_\pm - b)} + \frac{\alpha^2}{16(\lambda_\pm - a)^2} + \frac{\beta^2}{16(\lambda_\pm - b)^2} =: R_\pm. \tag{5.5}$$

The above expression for $\mu_\pm$ is the analogue of the spectral equation in the autonomous case, but here the spectral curve is not fixed since its coefficients vary with z; explicitly from (5.4) we have

$$H_a(z) = \frac{1}{2}\int^z (a-b)^{-1}(\lambda_+(y)-a)(\lambda_-(y)-a)\,\mathrm{d}y$$

and similarly for $H_b(z)$. Deformations of spectral curves have been used by several different authors to describe asymptotics of non-autonomous equations by the use of Whitham averaging or WKB-type methods (see [17] for a review). Instead we use the spectral equation (5.5) to derive an exact integral equation for the separation variables $\lambda_\pm$. From the expression for H in the elliptic coordinates we find

$$\lambda'_\pm = \frac{\partial H}{\partial \mu_\pm} = \frac{4(\lambda_\pm - a)(\lambda_\pm - b)\mu_\pm}{(\lambda_\mp - \lambda_\pm)}$$

which leads to a coupled integral equation for the vector $\lambda = (\lambda_+, \lambda_-)^T$:

$$\lambda(z) = \begin{pmatrix} I_+[\lambda](z) \\ I_-[\lambda](z) \end{pmatrix} =: I[\lambda](z)$$

with

$$I_\pm[\lambda](z) = 4\int_{z_0}^z \frac{(\lambda_\pm(s)-a)(\lambda_\pm(s)-b)}{(\lambda_\mp(s)-\lambda_\pm(s))}\sqrt{R_\pm(s)}\,\mathrm{d}s.$$

If the solution of (5.1) has a singularity at $z = z_0$, then locally (up to the symmetry $\lambda_+ \leftrightarrow \lambda_-$) the separation variables behave like

$$\lambda_+ \sim \frac{1}{(z-z_0)^2} \qquad \lambda_- \to c \qquad z \to z_0$$

for some constant c. In that case (e.g. by introducing a new variable $v = {\lambda_+}^{-1/2}$ analytic at z_0), in the neighbourhood of the singularity the operator $I[\lambda]$ can be converted into an integral operator defining a contraction mapping on a complete space of analytic functions, whose unique fixed point corresponds precisely to the solution of (5.1). Integral operators of this type were an important element in the direct proof [13] of the Painlevé property for the Painlevé equations PI–VI. The essential new feature here is that to treat higher-order equations or systems like (5.1) it is necessary to consider operators on vectors of analytic functions (i.e. with components v, λ_- in this case). A more detailed discussion of these integral operators will be given elsewhere [10].

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